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by Gino Cateau

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Abstract

This paper proposes a simple analytical method to determine the stationarity of an *unnormalized* variable from the solution to a normalized model i.e. a model whose variables must be expressed in relative terms or must be differenced for a solution to exist. The paper then applies the method to answer a question of interest to policy-makers: does optimal policy under commitment lead to stationarity in the price level? Unlike Gaspar, Smets, and Vestin (2007), the paper finds that optimal policy under commitment does not lead to price level stationarity in the Smets and Wouters (2003) model.

JEL classification: E52, E58
Bank classification: Monetary policy framework

Résumé

L'auteur propose une méthode analytique simple pour déterminer la stationnarité d'une variable *non normalisée* à partir de la solution d'un modèle normalisé – c'est-à-dire un modèle dont on a exprimé les variables en termes relatifs ou sous la forme de différences afin de pouvoir le résoudre. Il se sert de cette méthode pour répondre à une question importante pour les autorités monétaires : l'application d'une politique monétaire optimale permet-elle d'atteindre la stationnarité du niveau des prix? Contrairement à Gaspar, Smets et Vestin (2007), l'auteur conclut que la politique optimale ne conduit pas à la stationnarité du niveau des prix dans le modèle de Smets et Wouters (2003).

Classification JEL : E52, E58
Classification de la Banque : Cadre de politique monétaire

1. Introduction

Economists use dynamic models to analyze a variety of economic problems. Since they are typically interested in stationary solutions to their model, the model must be freed from unit roots for such a solution to exist (Blanchard and Khan 1980). Therefore, the model is normalized i.e. variables are expressed in relative terms or differenced prior to solving. However, for some purposes, it is important to determine the properties of the unnormalized variables. For instance, while models in the monetary policy literature are often cast in terms of the inflation rate, the policy-maker may want to know about the stationarity properties of the price level.

This paper proposes a simple analytical method to determine the stationarity of an unnormalized variable e.g. a level variable from the solution to a normalized model. Our approach is based on the mathematical definition of an impulse response. The basic intuition is the following: if a variable is stationary, temporary shocks must not have permanent effects on that variable. Our method uses the solution to the normalized model and the relationship between the normalized and unnormalized variable to derive a formula for the impulse response of the unnormalized variable to a temporary shock. It is easy to calculate and further, in the case of non-stationarity, it offers the added benefit of determining which shock, though temporary, has permanent effects on the unnormalized variable. By construction, the formula also calculates the magnitude of those permanent effects.

The paper then uses the formula to answer a question of interest to policy-makers: does optimal policy under commitment lead to stationarity in the price level. Woodford (2003) shows that price-level stationarity would indeed be a feature of optimal policy in a basic NKPC model. In fact, he derives the result assuming a quadratic loss function in inflation and the output gap. Therefore, even though the objective of monetary policy is not to stabilize the price level per se, price level stationarity still results. He however argues that that result was likely to be special to the basic NKPC model. He argues, for instance, that if the policy-maker also cared about stabilizing changes in the interest rate in his loss function, price level stationarity would not result.

Recently, Gaspar, Smets, and Vestin (2007), argued that price-level stationarity could still be a feature of optimal policy even in much more intricate models. Assuming a model with many more frictions than the basic NKPC (Smets and Wouters 2003) and a policy-maker that also cares about stabilizing interest rate changes in addition to inflation and output gap, they argue that the price level is stationary. Indeed, they produce figures of impulse responses of the price level to a temporary cost-push shock and argue that the effect of the shock eventually disappears.

In this paper, we revisit their analysis using our formula. We find that the price level is in fact not stationary: temporary shocks have permanent effects (although small) on the price level. Hence as remarked by Woodford (2003), price level stationarity is a feature of optimal policy under commitment only in special circumstances.

That result has implications for the inflation targeting versus price level targeting debate. Indeed, in view of its 2011 "renewal of the inflation control target" meetings with the Government of Canada, the Bank of Canada is currently seriously investigating the benefits and costs of switching from inflation targeting (which does not lead to price level stationarity) to price level targeting (which induces price level stationarity). What our result implies for that debate is that a switch to price level targeting should not be justified on the basis that optimal policy under commitment also leads to price level stationarity.[1] That conclusion is fragile and may not hold for more realistic models than the NKPC (e.g. Smets and Wouters 2003).

The paper is organized as follows: section 2 derives the formula for determining stationarity, section 3 applies the formula to determine stationarity in three different models, and section 4 concludes.

2. Determining stationarity of an unnormalized variable

Policy-makers in this paper determine optimal policy under commitment by minimizing a quadratic loss function

$$\min_{i_t} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ z_t' Q_z z_t + i_t' R i_t \right\}, \tag{1}$$

subject to a linear forward-looking model

$$\begin{aligned} H1_{zy} y_t + H1_{zz} z_{t-1} + H2_{zz} z_t + H3_{zz} E_t z_{t+1} + B_z i_t &= 0 \\ H1_{yy} y_t + H1_{yz} z_{t-1} + H2_{yy} y_{t+1} + C_y \epsilon_{t+1} &= 0 \end{aligned} \tag{2}$$

[1]Using an OLG model with aggregate uncertainty and nominal bonds, Kryvtsov, Shukayev, and Ueberfeldt (2007) find that optimal policy does not require price level stationarity. Optimal policy exhibits price level (and inflation) targeting but not price level stationarity.

where i_t is the policy-maker's policy instrument, z_t are endogenous state variables that are to be determined within the model once the policy-maker sets his instrument at time t, y_{t+1} are state variables over which the policy-maker has no control other than through the influence of past predetermined z_{t-1} and ϵ_{t+1} are the innovations of y_{t+1} for which $E_t(\epsilon_{t+1}) = 0$ and $E_t(\epsilon_{t+1}\epsilon'_{t+1}) = \Omega$.

First-order conditions to the above problem give rise to a difference equation in the state variables of the model, $X_t = \begin{bmatrix} y_{t+1} \\ z_t \end{bmatrix}$, and the associated co-state variables, μ_t (see appendix A). Therefore, solving for optimal policy involves solving the difference equation. The solution is known to exist under certain conditions (see Hansen and Sargent 2004, Blanchard and Khan 1980, Anderson and Moore 1985). Those conditions typically imply that the model is written in relative terms or rates rather than levels. For example, if the objective of the policy-maker is to control inflation, the model will be formulated such that X_t includes the inflation rate, π_t, rather than the price level p_t. If those conditions are satisfied, the optimal policy for i_t under commitment will be a function of the *predetermined* state and co-state variables at time t (see appendix A) i.e.

$$i_t = F_X X_{t-1} + F_\mu \mu_{t-1}, \tag{3}$$

The evolution of the economy is then given by

$$\begin{bmatrix} X_t \\ \mu_t \end{bmatrix} = N \begin{bmatrix} X_{t-1} \\ \mu_{t-1} \end{bmatrix} + C\epsilon_t. \tag{4}$$

and the transition matrix N has stable eigenvalues.

In a stationary system, the effect of a temporary shock must eventually disappear. This can be formalized through the impulse response function. Denoting $Z_t = \begin{bmatrix} X_t \\ \mu_t \end{bmatrix}$, the impulse response to a temporary shock that hits at time t_0

$$\frac{\partial Z_{t_0+\tau}}{\partial \epsilon_{t_0}} = N^\tau C \tag{5}$$

Hence if N has stable eigenvalues, the effect of a temporary shock that hits at time t_0 eventually disappears as τ tends to infinity.

But suppose that we were interested in the impulse response of the level of a variable.

To be more concrete, suppose that X_t is expressed in terms of the inflation rate, π_t but we are interested in the impulse response to the price level, p_t. We use the following approach to determine the stationarity of the price level: first, let h be a selection vector that picks out π_t from Z_t such that $\pi_t = hZ_t$. Then, since $p_t = p_{t-1} + \pi_t$,

$$\begin{aligned} p_{t+\tau} &= p_{t-1} + \sum_{j=0}^{\tau} \pi_{t+j} && (6) \\ &= p_{t-1} + \sum_{j=0}^{\tau} h \left\{ N^{j+1} Z_{t-1} + \sum_{k=0}^{j} N^{j-k} C \epsilon_{t+k} \right\} && (7) \end{aligned}$$

Therefore the effect of a temporary shock that hits at time t_0 on the price level can be gauged from the expression

$$\frac{\partial p_{t_0+\tau}}{\partial \epsilon_{t_0}} = h \sum_{j=0}^{\tau} N^j C \quad (8)$$

The price-level will be stationary if and only if the effect of any temporary shock eventually dies down i.e.

$$\begin{aligned} \lim_{\tau \uparrow \infty} \frac{\partial p_{t_0+\tau}}{\partial \epsilon_{t_0}} &= h(I-N)^{-1} C && (9) \\ &= 0. && (10) \end{aligned}$$

The last result will be our basis for determining whether the price level is stationary. Letting

$$r_\infty = h(I-N)^{-1} C, \quad (11)$$

given the matrices N and C that characterize the evolution of the economy in (A9), we will compute r_∞ and verify whether it yields a row of zeros. If it does, then that will imply that the effect of any shock on the price level eventually disappears i.e. the price level is stationary. If it does not, then some (or all) shocks have permanent effects on the price level.

3. Price level stationarity

3.1 The simple NKPC model

To illustrate how my approach works, I use as benchmark the simple NKPC in Woodford (2003). The model of the economy is

$$\pi_t - \gamma\pi_{t-1} = \beta(E_t\pi_{t+1} - \gamma\pi_t) + \kappa x_t + u_t \tag{12}$$

where π_t is the inflation rate, x_t is the output gap, $0 \leq \gamma \leq 1$ is the degree of indexation, and u_t is an exogenous cost-push shock following an AR(1) process

$$u_{t+1} = \rho_u u_t + \epsilon_{t+1}^u. \tag{13}$$

The policy-maker chooses the output gap to minimize

$$\sum_{t=0}^{\infty} \beta^t \left\{(\pi_t - \gamma\pi_{t-1})^2 + \omega x_t^2\right\}. \tag{14}$$

The above problem is a control problem that can be mapped into the typical linear-quadratic control problem defined in section 2. In fact, for the simple NKPC model, an analytical solution can even be obtained. It can be shown that under the full commitment solution, the optimal output gap will be

$$x_t = \frac{\kappa}{\omega}\left(-\frac{\varphi_1}{\beta(1-\varphi_1\rho_u)}u_t + \frac{1}{\varphi_2}\mu_{t-1}\right) \tag{15}$$

where $\varphi_1 < 1 < \varphi_2$, $\varphi_1 = \frac{2\beta}{1+\beta+\frac{\kappa^2}{\omega}+\sqrt{(1+\beta+\frac{\kappa^2}{\omega})^2-4\beta}}$, $\varphi_2 = \frac{2\beta}{1+\beta+\frac{\kappa^2}{\omega}-\sqrt{(1+\beta+\frac{\kappa^2}{\omega})^2-4\beta}}$, and μ_{t-1} is the co-state variable associated to the forward-looking variable.

The evolution of the economy is then given by

$$\begin{bmatrix} u_{t+1} \\ \pi_t \\ \mu_t \end{bmatrix} = \begin{bmatrix} \rho_u & 0 & 0 \\ \frac{\varphi_1}{\beta(1-\varphi_1\rho_u)} & \gamma & 1-\frac{1}{\varphi_2} \\ -\frac{\varphi_1}{\beta(1-\varphi_1\rho_u)} & 0 & \frac{1}{\varphi_2} \end{bmatrix} \begin{bmatrix} u_t \\ \pi_{t-1} \\ \mu_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \epsilon_{t+1}^u. \tag{16}$$

Letting N denote the transition matrix in (16), it is easy to show that the eigenvalues of N are respectively ρ_u, γ, and $\frac{1}{\varphi_2}$. Thus if the degree of indexation,γ, is less than 1, the eigenvalues of N are all stable.

Woodford (2003) shows that even though the loss function of policy-maker does not penalize variability in the absolute level of prices, the price level is stationary except when there is perfect indexation i.e. $\gamma = 1$. Given the evolution of the economy, (16), how can this be verified? Should we, for instance, use the identity linking the price level to inflation, $P_t = P_{t-1} + \pi_t$ to expand the system and determine stationarity of the resulting state vector $X_t' = [u_{t+1}, \pi_t, \mu_t, P_t]$? The answer is no: introducing the price level through the identity would automatically introduce a unit root in the system and hence invalidate the typical methods we use to determine stationarity of the expanded state vector (Hamilton 1994).

To determine stationarity of the price level, we can instead verify whether temporary shocks have temporary effects using (9).

Table 1: Price level stationarity in basic NKPC

γ	r_∞
0	0
0.5	0
1	49.12

Table 1 shows how r_∞ varies with $\gamma = 0, 0.5$ and 1. It confirms Woodford (2003)'s analysis. The effect of cost-push shock on the price level dies out for $0 \leq \gamma < 1$ but leads to a permanent increase in the price level for $\gamma = 1$ ($r_\infty = 49.12$). Why does this happen?

Figure 1 which shows the impulse responses of inflation, output gap and price level conveys the intuition. Consider the $\gamma = 0$ case, for instance. Since the policy-maker can control the output gap directly (the output gap is the instrument of the policy-maker in this example), following the unexpected inflationary cost-push shock, the policy-maker spreads the effect of the shock by reducing output below potential today but returning it to potential only gradually. Since rational forward-looking agents can anticipate this, the expectation of output being below potential for a while leads to expectations of lower prices for tomorrow. Hence, through the NKPC, (12), this also implies that the price increases today will be smaller than otherwise. As figure 1 illustrates, the fact that output is returned to potential gradually coupled with the smaller contemporaneous increase in prices means that inflation eventually undershoots its long run level (the $\gamma = 0$ curve drops below 0 after 2 quarters); the unexpected increase in prices is undone and the price-level becomes stationary.

Notice how a price level targeting regime would operate similar to the full-commitment

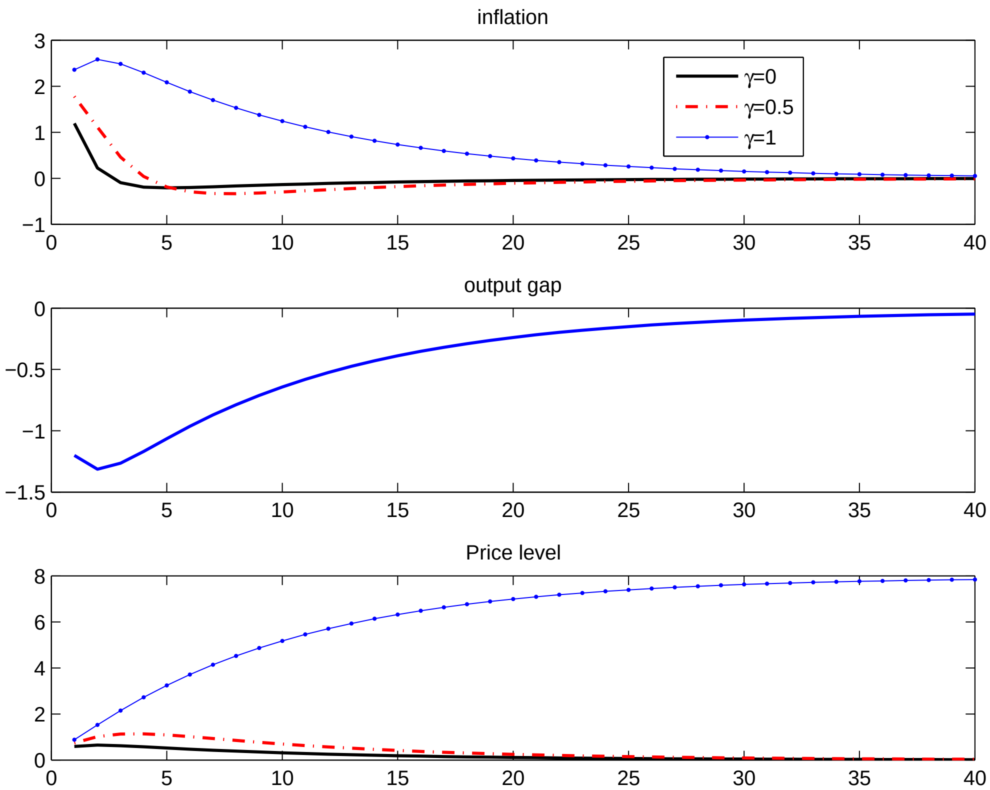

Figure 1: Impulse responses under different degrees of indexation

solution above. Indeed, under a price level targeting regime, in response to the unexpected increase in prices due to the cost-push shock, since agents expect the price level to eventually go back to target, they expect lower inflation for tomorrow. The lower expectations of inflation for tomorrow, through (12), imply that the policy-maker does not need to adjust the output gap by as much as he would have had to do had those expectations been unaffected. Hence output can be returned to potential gradually just as the full-commitment solution prescribes.

When $\gamma = 1$, the loss function (14) implies that the policy-maker is concerned with stabilizing the change in inflation rather than the inflation rate. In that case, no undershooting occurs. Inflation is gradually returned to its long run level after the cost-push shock. But since the shock is not undone, the price-level is non-stationary.

3.2 The NKPC and the IS curve

I now consider the case where the policy-maker's model consists of two equations: (i) the NKPC, (12), (ii) the IS curve

$$x_t = \mathrm{E}_t x_{t+1} - \sigma(i_t - \mathrm{E}_t \pi_{t+1} - r_t^n) \tag{17}$$

which relates the output gap to the nominal interest rate, expected inflation and the natural rate of interest, r_t^n. The natural rate of interest is an exogenous process following an AR(1),

$$r_{t+1}^n = \rho_r r_t^n + \epsilon_{t+1}^r \tag{18}$$

Since the policy-maker cannot directly control the output gap, the policy instrument of the policy-maker is now the nominal interest rate. The policy-maker chooses the interest rate to minimize

$$\sum_{t=0}^{\infty} \beta^t \left\{ (\pi_t - \gamma\pi_{t-1})^2 + \omega x_t^2 + \nu(i_t - i_{t-1})^2 \right\}. \tag{19}$$

In section 3.1, we showed that an important reason for price level stationarity was the fact that policy-maker had complete control on the output gap. In response to the temporary cost-push shock, the policy-maker could adjust the output gap gradually to undo the increase in prices following the shock. In the present case, since $\nu \neq 0$, the policy-maker does not have the luxury to move x_t as desired without caring about i_t. Therefore, owing to the policy-maker's preferences, there may be limits to how much the interest rate (and hence output gap) can adjust in response to shocks. Table 2 confirms that intuition. In addition

Table 2: Price level stationarity in NKPC and IS

ω	ν	r_∞ r_t^n	r_∞ u_t
0	0	0	0
0	0.5	-0.004	-0.53
0	1	-0.006	-0.69
0.5	0	0	0
0.5	0.5	-0.12	-0.29
0.5	1	-0.17	-0.39
1	0	0	0
1	0.5	-0.17	-0.48
1	1	-0.24	-0.66

to the degree of indexation being less than one, price level stationarity requires that ν, the weight that the policy-maker assigns to controlling changes in the interest rate, is zero.

Figure 2 shows the impulse responses to a natural rate shock and the cost-push shock when the degree of indexation is 0.5 and the weights to output gap and interest rate stabilization are both 0.5. Consider the natural rate shock. On impact, it leads to a fall in prices and an increase in the output gap. The policy-maker adjusts the interest rate but that adjustment never leads inflation to overshoot its long run level. Hence the initial impact of the shock is not undone; there is a permanent decline in the price level.

3.3 Smets and Wouters (2003)

Gaspar, Smets, and Vestin (2007) argue that optimal commitment policy induces price level stationarity in the Smets and Wouters (2003) model. Smets and Wouters (2003) is a much more elaborate model than the basic NKPC. It features three types of economic agents: households, firms and the central bank. Households decide how much to consume, how much to invest and how much to work and at what wage. Firms employ workers and capital and decide how much to produce and at what price to sell their products. In addition to a number of real frictions such as habit formation in consumption and adjustment costs in investment, the model features nominal price and wage rigidities. Thus, there are a number of frictions which make it costly to revert the price level. Yet, when analyzing the effect of a price mark-up shock, for instance, Gaspar, Smets, and Vestin (2007) point out that "...in spite of the other real and nominal frictions, optimal commitment policy again induces a

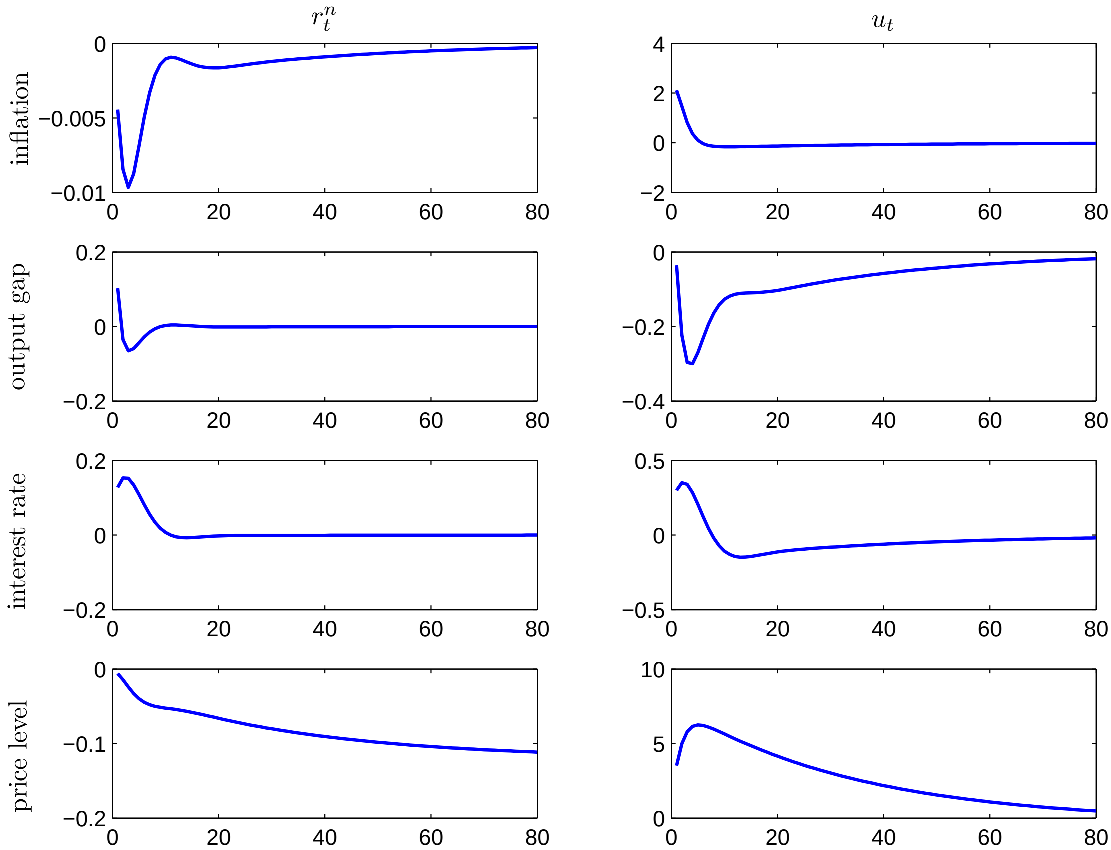

Figure 2: Impulse responses to natural rate and cost-push shock for the case $\omega = \nu = \gamma = 0.5$

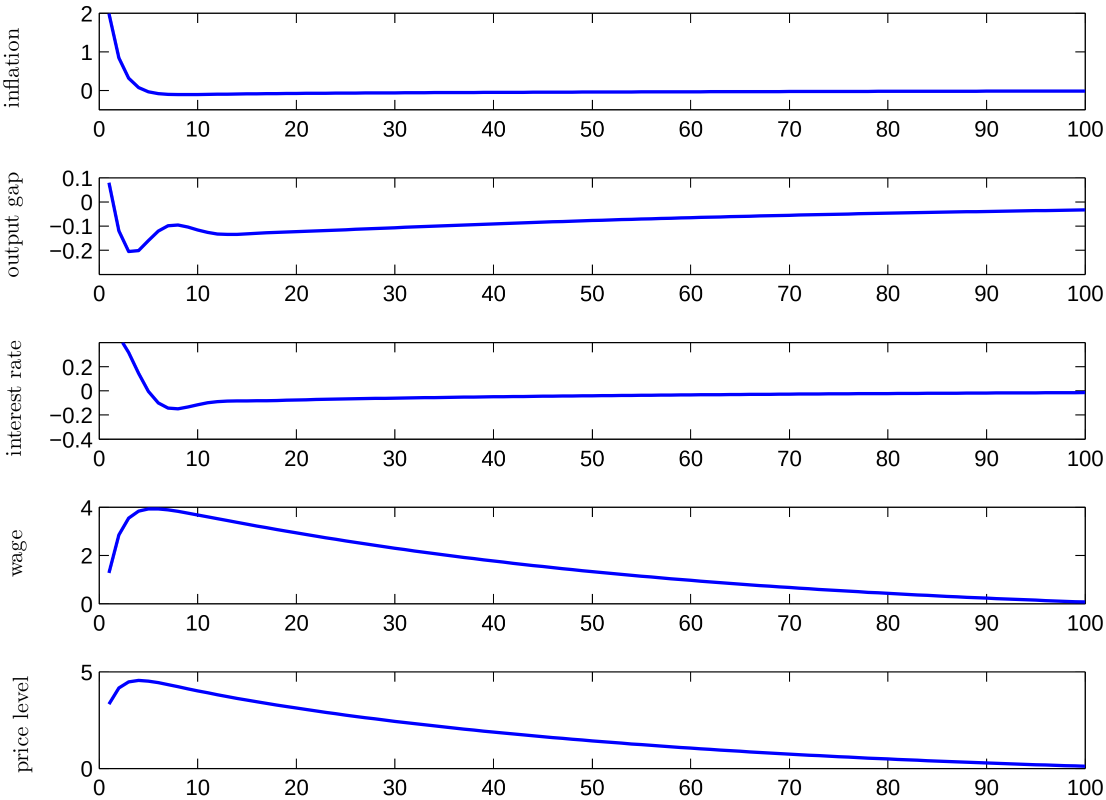

Figure 3: Impulse responses to price mark-up shock

stationary price level." (section 2.3, paragraph 2).

This section uses our method to verify that conclusion. As in Gaspar, Smets, and Vestin (2007), I assume a policy-maker that minimizes a quadratic loss function in the variability of the semi-difference of inflation, output gap and interest rate change with weights 0.9, 0.1 and 0.05 respectively under commitment. Figure 3 displays the impulse response of inflation, output gap, interest rate, wage, and price level to a 1 per cent price mark-up shock. The last panel shows that after leading to an increase in the price level of 5 per cent after 5 quarters, the effect of the price mark-up shock eventually diminishes. But is the shock to the price level fully reversed?

Table 3 reports the effect on the price level of a temporary 1 per cent shock. The rows of the first column displays the shock considered and the corresponding row in the second column reports r_∞. What the table tells us therefore is that the price level is not stationary. Each of the temporary shock lead to a permanent, albeit small, shift in the price level. Hence, what appears as price level stationarity in the last panel of figure 3 in fact is not.

Table 3: Price level stationarity in Smets and Wouters (2003).

shock	r_∞
govt. expenditure	0.0034
investment	0.0539
labor supply	-0.0005
productivity	-0.0005
preference	-0.0544
equity premium	-0.0029
price mark-up	0.1237
wage mark-up	0.0308

The effect of the shock diminishes over time but is never completely eliminated.

4. Conclusion

This paper proposes a simple analytical method to determine the stationarity of an *unnormalized* variable from the solution to a normalized model i.e. a model whose variables must be expressed in relative terms or must be differenced for a solution to exist. We use the solution to the normalized model to derive an explicit formula for the impulse response of the unnormalized variable to a temporary shock. Stationarity can then be determined by verifying whether a temporary shock has a temporary effect on the unnormalized variable.

The paper then applies the method to answer a question of interest to policy-makers: does optimal policy under commitment lead to stationarity in the price level? We use the formula to determine whether optimal policy under commitment leads to price level stationarity in three models: (i) the simple NKPC model in Woodford (2003), (ii) the NKPC model and an IS curve, and (iii) Smets and Wouters (2003). The paper first confirms Woodford's conclusion that optimal policy under commitment in the simple NKPC model policy leads to price level stationarity unless there is full indexation. Secondly, it confirms Woodford's intuition that the price level is in general not stationary if the policy-maker cares about stabilizing changes in the interest rate in the loss function. However, contrary to Gaspar, Smets, and Vestin (2007), the paper finds that the price level is non-stationary in Smets and Wouters (2003): temporary shocks have permanent effects (albeit small) on the price level.

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Appendix A: Full commitment solution

The full commitment solution is obtained by

$$\min_{i_t} \sum_{t=0}^{\infty} \beta^t \left\{ X_t' Q X_t + i_t' R i_t \right\}, \tag{A1}$$

subject to the model

$$H_1 X_{t-1} + H_2 X_t + H_3 X_{t+1} + B i_t = 0. \tag{A2}$$

Notice that since the loss function (A1) is quadratic and the model linear, I can solve the non-stochastic version of the policy-maker's problem owing to certainty equivalence.

The Lagrangian for this problem is

$$L = \sum_{t=0}^{\infty} \beta^t \left\{ X_t' Q X_t + i_t' R i_t + 2\mu_t' \left(H_1 X_{t-1} + H_2 X_t + H_3 X_{t+1} + B i_t \right) \right\}. \tag{A3}$$

The first-order conditions are

$$i_t \quad : \quad i_t = -R^{-1} B' \mu_t \tag{A4}$$

$$X_t \quad : \quad Q X_t + H_1' \beta \mu_{t+1} + H_2' \mu_t + H_3' \beta^{-1} \mu_{t-1} = 0. \tag{A5}$$

By substituting the f.o.c.'s for i_t into the constraint (A2), we obtain

$$H_1 X_{t-1} + H_2 X_t + H_3 X_{t+1} - B R^{-1} B' \mu_t = 0. \tag{A6}$$

From (A6) and (A5), I can construct a system of difference equations in X_t and μ_t :

$$\begin{aligned} \begin{bmatrix} H_1 & 0 \\ 0 & H_3' \beta^{-1} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ \mu_{t-1} \end{bmatrix} & + \begin{bmatrix} H_2 & -B R^{-1} B' \\ Q & H_2' \end{bmatrix} \begin{bmatrix} X_t \\ \mu_t \end{bmatrix} \\ & + \begin{bmatrix} H_3 & 0 \\ 0 & H_1' \beta \end{bmatrix} \begin{bmatrix} X_{t+1} \\ \mu_{t+1} \end{bmatrix} = 0, \end{aligned} \tag{A7}$$

which can be rewritten as

$$A_1 \begin{bmatrix} X_{t-1} \\ \mu_{t-1} \end{bmatrix} + A_2 \begin{bmatrix} X_t \\ \mu_t \end{bmatrix} + A_3 \begin{bmatrix} X_{t+1} \\ \mu_{t+1} \end{bmatrix} = 0. \tag{A8}$$

It can be shown that given the transversality conditions and appropriate initial conditions X_{-1} and μ_{-1}, the solution to the difference equation (A8) is

$$\begin{bmatrix} X_t \\ \mu_t \end{bmatrix} = N \begin{bmatrix} X_{t-1} \\ \mu_{t-1} \end{bmatrix}. \tag{A9}$$

The matrix N can be solved for using invariant subspace methods (e.g. Dennis 2003) or iterative methods. The full commitment decision rule for i_t is then obtained from (A4) and (A9). From (A9),

$$\mu_t = \begin{bmatrix} 0 & I \end{bmatrix} N \begin{bmatrix} X_{t-1} \\ \mu_{t-1} \end{bmatrix}. \tag{A10}$$

From (A4), it follows that

$$i_t = -R^{-1}B' \begin{bmatrix} 0 & I \end{bmatrix} N \begin{bmatrix} X_{t-1} \\ \mu_{t-1} \end{bmatrix}. \tag{A11}$$

I will write the full commitment solution as $i_t = F_X X_{t-1} + F_\mu \mu_{t-1}$.

A.1 Dynamics in a stochastic system

In this section I consider the problem recast as a stochastic system. Beginning with

$$H_1 X_{t-1} + H_2 X_t + H_3 X_{t+1} + B i_t + C\epsilon_{t+1} = 0, \tag{A12}$$

and performing similar substitutions and manipulations as in section ??, I obtain the difference system

$$A_1 \begin{bmatrix} X_{t-1} \\ \mu_{t-1} \end{bmatrix} + A_2 \begin{bmatrix} X_t \\ \mu_t \end{bmatrix} + A_3 \begin{bmatrix} X_{t+1} \\ \mu_{t+1} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \epsilon_{t+1} = 0. \tag{A13}$$

Using (A9), I get

$$\begin{bmatrix} X_t \\ \mu_t \end{bmatrix} = N \begin{bmatrix} X_{t-1} \\ \mu_{t-1} \end{bmatrix} + D\epsilon_{t+1}. \tag{A14}$$

where $D = (A_2 + A_3 N)^{-1} \begin{bmatrix} C \\ 0 \end{bmatrix}$.

Appendix B: Smets and Wouters (2003)

I include below the equations of Smets and Wouters (2003) and the calibration of the parameters in table B. I used Uhlig (2006) version and calibration of Smets and Wouters (2003).

The capital accumulation equation:

$$\hat{K}_t = (1-\tau)\hat{K}_{t-1} + \tau\hat{I}_{t-1} \tag{B1}$$

The labour demand equation:

$$\hat{L}_t = -\hat{w}_t + (1+\psi)\hat{r}_t^k + \hat{K}_{t-1} \tag{B2}$$

The goods market equilibrium condition:

$$\hat{Y}_t = (1 - \tau k_y - g_y)\hat{C}_t + \tau k_y \hat{I}_t + \epsilon_t^G \tag{B3}$$

The production function:

$$\hat{Y}_t = \phi\epsilon_t^a + \phi\alpha\hat{K}_{t-1} + \phi\alpha\psi\hat{r}_t^k + \phi(1-\alpha)\hat{L}_t \tag{B4}$$

The monetary policy reaction function, a Taylor-type rule:

$$\begin{aligned}\hat{R}_t &= \rho\hat{R}_{t-1} + (1-\rho)\Big\{\bar{\pi}_t + r_\pi(\hat{\pi}_{t-1} - \bar{\pi}_t) + r_Y(\hat{Y}_t - \hat{Y}_t^P)\Big\} \\ &\quad + r_{\Delta\pi}(\hat{\pi}_t - \hat{\pi}_{t-1}) + r_{\Delta Y}\Big(\hat{Y}_t - \hat{Y}_t^P - (\hat{Y}_{t-1} - \hat{Y}_{t-1}^P)\Big),\end{aligned} \tag{B5}$$

where $\hat{Y}_t^P$ refers to a hypothetical "frictionless economy" and **potential output**. The difference $\hat{Y}^t - \hat{Y}_t^P$ is the **output gap**.

The consumption equation:

$$\begin{aligned}\hat{C}_t &= \frac{h}{1+h}\hat{C}_{t-1} + \frac{h}{1+h}E_t\hat{C}_{t+1} - \frac{1-h}{(1+h)\sigma_c}(\hat{R}_t - Et\hat{\pi}_{t+1}) \\ &\quad + \frac{1-h}{(1+h)\sigma_c}\hat{\epsilon}_t^b \end{aligned} \tag{B6}$$

The investment equation:

$$\hat{I}_t = \frac{1}{1+\beta}\hat{I}_{t-1} + \frac{\beta}{1+\beta}E_t\hat{I}_{t+1} + \frac{\varphi}{1+\beta}\hat{Q}_t + \hat{\epsilon}_t^I \tag{B7}$$

The Q equation:

$$\hat{Q}_t = -(\hat{R}_t - E_t\hat{\pi}_{t+1}) + \frac{1-\tau}{1-\tau+\bar{r}^k}E_t\hat{Q}_{t+1} + \frac{\bar{r}^k}{1-\tau+\bar{r}^k}E_t\hat{r}_{t+1}^k + \eta_t^Q \tag{B8}$$

The inflation equation:

$$\begin{aligned}\hat{\pi}_t &= \frac{\beta}{1+\beta\gamma_p}E_t\hat{\pi}_{t+1} + \frac{\gamma_p}{1+\beta\gamma_p}\hat{\pi}_{t-1} \\ &\quad \frac{1}{1+\beta\gamma_p}\frac{(1-\beta\xi_p)(1-\xi_p)}{\xi_p}\left[\alpha\hat{r}_t^k + (1-\alpha)\hat{w}_t - \hat{\epsilon}^a\right] + \eta_t^p \end{aligned} \tag{B9}$$

The wage equation:

Table B1: Calibration

Parameter	Value	Description
β	0.99	Discount factor
τ	0.025	Depreciation rate of capital
α	0.3	Capital output ratio
ψ	$\frac{1}{0.169}$	Inverse elasticity of capital utility cost
γ_p	0.469	Degree of partial indexation of price
γ_w	0.763	Degree of partial indexation of wage
λ_w	0.5	Mark-up in wage setting
ξ_s^p	0.908	Calvo price stickiness
ξ_s^w	0.737	Calvo wage stickiness
σ_L	2.4	Inverse elasticity of labour supply
σ_c	1.353	Coefficient of relative risk aversion
h	0.573	Habit portion of past consumption
ϕ	1.408	1 plus share of fixed costs in production
φ	$\frac{1}{6.771}$	Inverse of inventory adjustment cost
$\bar{r}_k$	$\frac{1}{\beta} - 1 + \tau$	Steady-state return on capital
k_y	8.8	Capital-output ratio
inv_y	0.22	Investment share in GDP
c_y	0.6	Consumption share in GDP
k_y	$\frac{inv_y}{\tau}$	Capital income share
g_y	$1 - c_y - inv_y$	Government expenditure share in GDP;
$r_{\Delta\pi}$	0.14	Inflation growth coefficient
r_y	0.099	Output gap coefficient
$r_{\Delta Y}$	0.159	Output gap growth coefficient
r_π	1.684	Inflation coefficient

$$\begin{aligned}
\hat{w}_t \;=\;& \frac{\beta}{1+\beta}E_t\hat{w}_{t+1} + \frac{1}{1+\beta}\hat{w}_{t-1} + \frac{\beta}{1+\beta}E_t\hat{\pi}_{t+1} \\
& - \frac{1+\beta\gamma_w}{1+\beta}\hat{\pi}_t + \frac{\gamma_w}{1+\beta}\hat{\pi}_{t-1} \\
& - \frac{1}{1+\beta}\frac{(1-\beta\xi_w)(1-\xi_w)}{\left(1+\frac{(1+\lambda_w)\sigma_L}{\lambda_w}\right)\xi_w} * \cdots \qquad \text{(B10)} \\
& \left[\hat{w}_t - \sigma_L\hat{L}_t - \frac{\sigma_c}{1-h}(\hat{C}_t - h\hat{C}_{t-1}) + \hat{\epsilon}_t^L\right] + \eta_t^w \qquad \text{(B11)}
\end{aligned}$$

Table B2: Calibration, continued

Parameter	Value	Description
ρ	0.961	AR for lagged interest rate
ρ_{ϵ_L}	0.889	AR for labour supply shock
ρ_{ϵ_a}	0.823	AR for productivity shock
ρ_{ϵ_b}	0.855	AR for preference shock
ρ_G	0.949	AR for government expenditure shock
$\rho_{\bar{\pi}}$	0.924	AR for inflation objective shock
ρ_{ϵ_i}	0.927	AR for investment shock
ρ_{ϵ_r}	0	AR for interest rate shock, IID
ρ_{λ_w}	0	AR for wage markup, IID
ρ_q	0	AR for return on equity, IID
ρ_{λ_q}	0	AR for price mark-up shock, IID
σ_{ϵ_L}	3.52	Standard deviation of labour supply shock
σ_{ϵ_a}	0.598	Standard deviation of productivity shock
σ_{ϵ_b}	0.336	Standard deviation of preference shock
σ_G	0.325	Standard deviation of government expenditure shock
$\sigma_{\bar{\pi}}$	0.017	Standard deviation of inflation objective shock
σ_{ϵ_r}	0.081	Standard deviation of interest rate shock
σ_{ϵ_i}	0.085	Standard deviation of investment shock
σ_{λ_p}	0.16	Standard deviation of mark-up shock
σ_{λ_w}	0.289	Standard deviation of wage mark-up shock
σ_{ϵ_q}	0.604	Standard deviation of equity premium shock.